Int. J. Solids Structures Vol. 23, No. 7, pp. 821–840, 1987
Printed in Great Britain.
0020-7683/87 $3.00+ .00

STRAIN- AND STRESS-BASED CONTINUUM DAMAGE MODELS—I. FORMULATION

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(*Received* 4 *April* 1986; *in revised form* 8 *September* 1986)

Abstract—Continuum elastoplastic damage models employing irreversible thermodynamics and internal state variables are developed within two alternative *dual* frameworks. In a *strain* [*stress*]-based formulation, damage is characterized through the *effective stress* [*strain*] concept together with the *hypothesis of strain* [*stress*] *equivalence*, and plastic flow is introduced by means of an *additive split of the stress* [*strain*] *tensor*. In a *strain*-based formulation we redefine the *equivalent strain*, usually defined as the J_2-norm of the strain tensor, as the (undamaged) energy norm of the *strain* tensor. In a *stress*-based approach we employ the complementary energy norm of the *stress* tensor. These thermodynamically motivated definitions result, for ductile damage, in symmetric elastic-damage moduli. For brittle damage, a simple *strain*-based anisotropic characterization of damage is proposed that can predict crack development parallel to the axis of loading (splitting mode). The strain- and stress-based frameworks lead to *dual* but not equivalent formulations, neither physically nor computationally. A viscous regularization of strain-based, rate-independent damage models is also developed, with a structure analogous to viscoplasticity of the Perzyna type, which produces retardation of microcrack growth at higher strain rates. This regularization leads to well-posed initial value problems. Application is made to the cap model with an isotropic strain-based damage mechanism. Comparisons with experimental results and numerical simulations are undertaken in Part II of this work.

1. INTRODUCTION

"Continuous damage mechanics" (CDM) have been introduced and employed extensively to describe the progressive degradation experienced by the mechanical properties of materials prior to the initiation of macrocracks. Kachanov[1] was the first to introduce the *effective stress* concept to model creep rupture. Later, damage mechanics were developed to model fatigue[2, 3], creep[4–7], creep–fatigue interaction[8, 9], and ductile plastic damage[10–15]. Recently, the CDM was applied to brittle materials[16, 17] such as concrete[18–22] and rock[23, 24].

Continuum damage theories are based on the thermodynamics of irreversible processes[6, 16, 25–28] and the internal state variable theory. To model isotropic damage processes it suffices to consider a scalar damage variable[22, 29], whereas tensor valued damage variables (second or fourth order) are required in order to account for anisotropic damage[26, 30–33]. Isotropic damage formulations are extensively employed in the literature because of their simplicity, efficiency, and adequacy for many practical applications.

In this paper, the first part of a sequence of two, we develop damage models within two possible alternative frameworks, either strain or stress based, which are capable of accommodating non-linear elastic response and general plastic response. Basic features of *strain space* formulations are the following. (a) For ductile damage, the *equivalent strain* concept, usually defined as the J_2 invariant of the strain tensor[13, 21, 22, 29], is redefined here as the (*undamaged*) *energy norm* of the strain tensor. For brittle damage, as a natural extension, we consider the energy norm associated with the *positive part* of the strain tensor; i.e. the projection of the strain tensor associated with positive eigenvalues. (b) Damage is introduced through the notion of *effective stress* and the *hypothesis of strain equivalence*. (c) Plastic response is formulated in effective stress space through an *additive split* of the *stress tensor*. By formulating plastic response in terms of effective stresses one effectively obtains a reduction, with increasing damage, in the material constants characterizing plastic flow.

In a *stress space* formulation, on the other hand, (a) the notion of *equivalent strain* is defined as the *complementary energy norm* of the stress tensor, (b) damage is characterized through the notion of *effective strain* together with the hypothesis of *stress equivalence* and (c) plastic response is characterized by means of an additive split of the *strain tensor*. For the ductile damage case it is shown that these definitions lead, for both strain- and stress-based models, to *symmetric* elastic-damage tangent moduli. We note that this is not the case for the usual J_2 characterization of equivalent strain. An outline of the paper is as follows. In Section 3 we develop the framework for strain-based damage models with attention restricted to isotropy. We conclude this section with the development of a rate-dependent (viscous) damage model that produces retardation of microcracking at higher strain rates, in agreement with some experimental results. This model is a viscous regularization of rate-independent damage, with a structure analogous to that of viscoplasticity of the Perzyna type. This model satisfies the *positiveness* condition of Valanis[34].

In Section 4, we extend the ideas of Section 3 and develop a simple *strain-based anisotropic* damage model for initially linear materials. The main idea is to treat the stiffness moduli as (tensorial) internal state variables. As a result, for brittle materials, failure modes in uniaxial compression with cracking development parallel to the axis of loading, the so-called "splitting modes", can be predicted within the present phenomenological context. Section 5 is concerned with the development of the alternative stress-based framework for damage models.

An essential characteristic of the proposed strain- and stress-based damage models is the remarkable simplicity of their numerical implementation in the context of finite element or finite difference methods, leading to a methodology ideally suited for large-scale computation. These and related computational issues are considered in detail in Part II of this work.

Development of suitable strain localization limiters that prevent mesh-sensitiveness associated with strain-softening is, currently, an active area of research that is not addressed in this paper. It is felt, however, that the present formulation can be readily modified to include several recently developed approaches; for instance, non-local damage equations of evolution, as in Bazant and Belytschko[57] or modified softening parameters as in Refs [58, 59]. These issues will be addressed in a forthcoming publication.

2. CONTINUUM DAMAGE: ALTERNATIVE FRAMEWORKS

We shall review some basic concepts of continuum damage mechanics needed for subsequent developments of two possible alternative frameworks: *strain*-space damage models, based on the notion of *effective stress* and considered in Sections 3 and 4; and *stress*-space damage models based on the *effective strain* concept and considered in Section 5.

2.1. *Effective stress concept and hypothesis of strain equivalence*

Physically, degradation of the material properties is the result of the initiation, growth and coalescence of microcracks or microvoids. Within the context of continuum mechanics, one may model this process by introducing an internal damage variable which can be a scalar or a tensorial quantity. We denote by $\mathbf{M}$ a fourth-order tensor which characterizes the state of damage and transforms the homogenized stress tensor $\boldsymbol{\sigma}$ into the effective stress tensor $\bar{\boldsymbol{\sigma}}$ (or vice versa); explicitly

$$\bar{\boldsymbol{\sigma}} := \mathbf{M}^{-1} : \boldsymbol{\sigma}. \tag{1}$$

For the isotropic damage case, the mechanical behavior of microcracks or microvoids is independent of their orientation and depends only on a scalar variable d. Accordingly, $\mathbf{M}$ will simply reduce to $(1-d)\mathbf{I}$, where $\mathbf{I}$ is the rank four identity tensor, and eqn (1) collapses to

$$\boldsymbol{\sigma}(t) := \frac{\boldsymbol{\sigma}(t)}{1-d(t)} \tag{2}$$

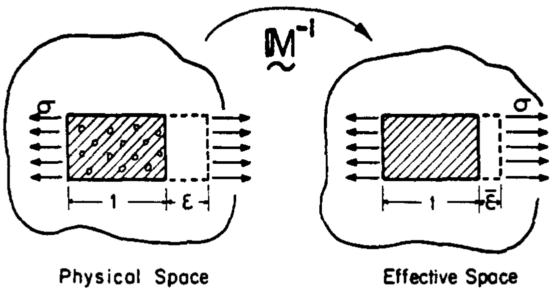

Fig. 1. Schematic illustration of the *hypothesis of strain.*

where $d(t) \in [0, d_c]$ for $t \in \mathbb{R}_+$, is the damage parameter, $\boldsymbol{\sigma}(t)$ the Cauchy stress tensor, and $\bar{\boldsymbol{\sigma}}(t)$ is the effective stress tensor, both at time t. Here, $d_c \in (0, 1]$ is a given constant. The coefficient $1 - d(t)$ dividing the stress tensor in eqn (2) is a *reduction factor* associated with the amount of damage in the material first introduced by Kachanov[1]. The value $d = 0$ corresponds to the *undamaged* state whereas a value $d \in (0, d_c)$ corresponds to a *damaged* state. The value $d = d_c$ defines *complete local rupture*[13, 29]. The damage parameter d may be interpreted physically as the ratio of damaged surface area over total (nominal) surface area at a local material point. In addition, Lemaitre[3, 27] introduced the following *hypothesis of strain equivalence*:

> "the strain associated with a damaged state under the applied stress is equivalent to the strain associated with its undamaged state under the effective stress".

See Fig. 1 for a schematic explanation.

2.2. *Effective strain concept and hypothesis of stress equivalence*

As an alternative to the concept of effective stress, we may consider the following notion of *effective strain*[33]

$$\bar{\boldsymbol{\varepsilon}}(t) := \mathbf{M} : \boldsymbol{\varepsilon}(t) \qquad (anisotropic) \tag{3}$$

$$\bar{\boldsymbol{\varepsilon}}(t) := [1 - d(t)]\boldsymbol{\varepsilon}(t) \qquad (isotropic). \tag{4}$$

Here, $\boldsymbol{\varepsilon}(t)$ is the strain tensor and $\bar{\boldsymbol{\varepsilon}}(t)$ is the effective strain tensor. By analogy with the hypothesis of strain equivalence and invoking similar homogenization techniques, we propose the following dual *hypothesis of stress equivalence*:

> "the stress associated with a damaged state under the applied strain is equivalent to the stress associated with its undamaged state under the effective strain".

We refer to Fig. 2 for a schematic illustration of this hypothesis.

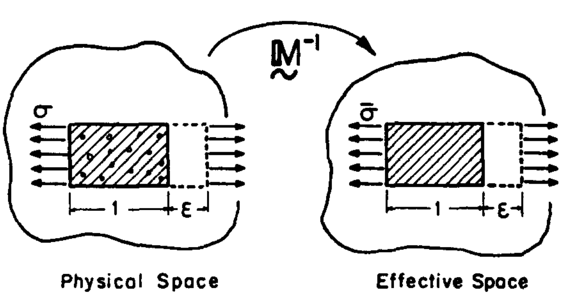

Fig. 2. Schematic illustration of the *hypothesis of stress equivalence.*

Remark 2.1. The effective stress concept and the hypothesis of strain equivalence are naturally associated with a *strain-based* formulation of elastoplastic-damage constitutive equations. Alternatively, the effective strain concept along with the hypothesis of stress equivalence correspond to a *stress-based* formulation of the elastoplastic-damage constitutive equations. Our thermodynamic framework will clarify further this basic distinction. □

3. A STRAIN-BASED ISOTROPIC CONTINUUM DAMAGE MODEL

The crucial idea underlining the strain-based isotropic continuum damage model presented in this section is the hypothesis that damage in the material is directly linked to the history of total strains. The notion of effective stress along with the hypothesis of strain equivalence then follow from the assumed form of free energy. Attention is focused on isotropic damage. The extension of the ideas presented in this section to the anisotropic (brittle) damage case is considered in Section 4.

3.1. *Thermodynamic basis. Stress split*

To introduce both damage and plastic flow processes, we consider a free energy potential of the following form:

$$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{\mathrm{p}}, \mathbf{q}, d) := (1-d)\Psi^0(\boldsymbol{\varepsilon}) - \boldsymbol{\varepsilon} : \boldsymbol{\sigma}^{\mathrm{p}} + \Xi(\mathbf{q}, \boldsymbol{\sigma}^{\mathrm{p}}) \tag{5}$$

where $d \in [0, d_c]$ is the *damage parameter*, $\mathbf{q}$ a suitable set of internal (*plastic*) variables, $\boldsymbol{\sigma}$ the strain tensor, and $\boldsymbol{\sigma}^{\mathrm{p}}$ the *plastic relaxation* stress tensor. In addition, $\Xi(\mathbf{q}, \boldsymbol{\sigma}^{\mathrm{p}})$ denotes a plastic potential function and $\Psi^0(\boldsymbol{\varepsilon})$ is the *initial elastic* stored energy function of the undamaged (virgin) material. We recall that $\boldsymbol{\varepsilon} \to \Psi^0(\boldsymbol{\varepsilon})$ is a *convex function*† in the space S of symmetric rank-2 tensors. In particular, for the linear case we have $\Psi^0(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C}^0 : \boldsymbol{\varepsilon}$ where $\mathbf{C}^0$ denotes the linear elasticity tensor.

Within the present strain space framework we introduce plastic flow by means of an additive split of the stress tensor into *initial* and *inelastic parts* that follows from the assumed structure of the free energy. Confining our attention to the purely mechanical theory, the Clausius–Duhem (reduced dissipation) inequality[35] takes the form

$$-\dot{\Psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geqslant 0 \tag{6}$$

for any admissible process. By taking the time derivative of eqn (5), substituting into eqn (6), and making use of standard arguments[36, 37] along with the additional assumption that damage and plastic unloading are elastic processes (in agreement with the characterizations discussed below), we obtain‡

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = (1-d)\frac{\partial \Psi^0}{\partial \boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^{\mathrm{p}} \tag{7}$$

and the dissipative inequalities

$$\Psi^0(\boldsymbol{\varepsilon})\dot{d} \geqslant 0 \quad \text{and} \quad -\frac{\partial \Xi}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} - \left(\frac{\partial \Xi}{\partial \boldsymbol{\sigma}^{\mathrm{p}}}\right) : \dot{\boldsymbol{\sigma}}^{\mathrm{p}} \geqslant 0. \tag{8}$$

It follows from eqn (7) that within the present strain space formulation, the stress tensor is split into *elastic-damage* and *plastic relaxation* parts. It is also clear from eqns (7) and (8) that the present framework is capable of accommodating general (nonlinear) elastic response and general plastic response.

†This means that $\Psi^0(\alpha\boldsymbol{\varepsilon}_1 + (1-\alpha)\boldsymbol{\varepsilon}_2) \leqslant \alpha\Psi^0(\boldsymbol{\varepsilon}_1) + (1-\alpha)\Psi^0(\boldsymbol{\varepsilon}_2)$, where $\alpha \in [0, 1]$.

‡The process leading to eqn (7) by exploiting the Clausius–Duhem inequality is often referred to as *Coleman's method.*

Remark 3.1. The potential $\Xi(\mathbf{q}, \boldsymbol{\sigma}^p)$ is linked to plastic dissipation. Its role is such that inequality $(8)_2$ is satisfied for arbitrary processes. Note that we have assumed Ξ independent of d. From eqn (5) it then follows that

$$-Y := -\frac{\partial\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, \mathbf{q}, d)}{\partial d} = \Psi^0(\boldsymbol{\varepsilon}). \tag{9}$$

Hence, the *initial* (*undamaged*) *elastic* strain energy $\Psi^0(\boldsymbol{\varepsilon})$ is the thermodynamic force—Y *conjugate* to the damage variable d. □

Remark 3.2. Within the framework outlined above, the physically *relevant notion* of *plastic strain*, $\boldsymbol{\varepsilon}^p$, is formulated in terms of *local unloading* as follows. Let $\boldsymbol{\Gamma}(\boldsymbol{\varepsilon}) := \partial\Psi^0(\boldsymbol{\varepsilon})/\partial\boldsymbol{\varepsilon}$. By definition, $\boldsymbol{\varepsilon}^p$ is the residual strain after unloading with $\boldsymbol{\sigma}^p$ held *fixed*. Hence, assuming $\boldsymbol{\Gamma}$ invertible, $\boldsymbol{\varepsilon}^p$ satisfies

$$\mathbf{0} = (1-d)\boldsymbol{\Gamma}(\boldsymbol{\varepsilon}^p) - \boldsymbol{\sigma}^p \Rightarrow \boldsymbol{\varepsilon}^p = \boldsymbol{\Gamma}^{-1}\left(\frac{\boldsymbol{\sigma}^p}{1-d}\right). \tag{10}$$

Note that in the absence of damage, the stress and strain splits are equivalent for $\boldsymbol{\Gamma}$ linear. □

3.2. *Strain-based characterization of damage. Elastic-damage moduli*

We first characterize the progressive degradation of mechanical properties of the material due to damage by means of a simple isotropic damage mechanism. To this end, we make use of the notion of *equivalent strain* $\bar{\tau}$. Motivated by Remark 3.1, we propose to define $\bar{\tau}$ as the (undamaged) *energy norm* of the strain tensor. This definition is at variance with that employed by Mazars and Lemaitre[13, 22] as the J_2-norm of the strain tensor. It will be shown that the former leads to symmetric elastic-damage moduli whereas the latter results in lack of symmetry. Accordingly, we set

$$\bar{\tau} := \sqrt{(2\Psi^0(\boldsymbol{\varepsilon}))}. \tag{11}$$

We then characterize the state of damage in the material by means of a damage criterion $g(\bar{\tau}_t, r_t) \leqslant 0$, formulated in strain space, with the following functional form:

$$g(\bar{\tau}_t, r_t) := \bar{\tau}_t - r_t \leqslant 0, \qquad t \in \mathbb{R}_+. \tag{12}$$

Here, the subscript t refers to value at current time $t \in \mathbb{R}_+$, and r_t is the damage threshold at current time t. If r_0 denotes the initial damage threshold before any loading is applied, a property characteristic of the material, we must have that $r_t \geqslant r_0$. Condition (12) then states that damage in the material is initiated when *the energy norm* of the strain tensor, $\bar{\tau}_t$ exceeds the initial damage threshold r_0. For the isotropic case, we define the evolution of the damage variable d by the rate equation

$$\begin{aligned} \dot{d}_t &= \dot{\mu} H(\bar{\tau}_t, d_t) \\ \dot{r} &= \dot{\mu} \end{aligned} \tag{13a}$$

where $\dot{\mu} \geqslant 0$ is a *damage consistency* parameter that defines *damage loading/unloading* conditions according to the Kuhn–Tucker relations

$$\dot{\mu} \geqslant 0, \qquad g(\bar{\tau}_t, r_t) \leqslant 0, \qquad \dot{\mu} g(\bar{\tau}_t, r_t) = 0. \tag{13b}$$

Conditions (13b) are standard for problems involving unilateral constraint. If $g(\bar{\tau}_t, r_t) < 0$, the damage criterion is not satisfied and by condition $(13b)_3$ $\dot{\mu} = 0$; hence, the damage rule (13a) implies that $\dot{d} = 0$ and no further damage takes place. If, on the other hand, $\dot{\mu} > 0$; that is, further damage ("loading") is taking place, condition $(13b)_3$ now implies that $g(\bar{\tau}_t, r_t) = 0$. In this event the value of $\dot{\mu}$ is determined by the damage consistency condition; i.e.

$$g(\bar{\tau}_t, r_t) = \dot{g}(\bar{\tau}_t, r_t) = 0 \Rightarrow \dot{\mu} = \dot{\bar{\tau}}_t. \tag{14a}$$

So that r_t is given by the expression

$$r_t = \max\left\{r_0, \max_{s \in (-\infty, t]} \bar{\tau}_s\right\}. \tag{14b}$$

Remark 3.3. If $H(d_t, \bar{\tau}_t)$ in condition (13a) is independent of d_t, the above formulation may be rephrased as follows. Let $G\colon \mathbb{R} \to \mathbb{R}_+$ be such that $H(\bar{\tau}_t) =: \partial G(\bar{\tau}_t)/\partial \bar{\tau}_t$. We shall assume that $G(\cdot)$ is *monotonic*. A damage criterion entirely equivalent to conditions (12) is given by $\bar{g}(\bar{\tau}_t, r_t) := G(\bar{\tau}_t) - G(r_t) \leqslant 0$. The flow rule (13a) and loading/unloading conditions (13b) then become

$$\begin{aligned} \dot{d}_t &= \dot{\mu}\frac{\partial \bar{g}(\bar{\tau}_t, r_t)}{\partial \bar{\tau}_t}, \qquad \dot{r}_t = \dot{\mu} \\ \dot{\mu} &\geqslant 0, \qquad \bar{g}(\bar{\tau}_t, r_t) \leqslant 0, \qquad \dot{\mu}\bar{g}(\bar{\tau}_t, r_t) = 0. \end{aligned} \tag{15}$$

In Section 4 it will be shown that conditions (15) are simply the Kuhn–Tucker optimality conditions of a *principle of maximum damage dissipation*. This interpretation is essential for the variational formulation considered in Part II of this paper. □

Remark 3.4. It should be noted that for the elastic-damage case r_0 does *not* correspond to the peak of the stress–strain curve in the uniaxial test. This is at variance with Ref. [21]. □

3.2.1. *Elastic-damage tangent moduli.* For ductile damage, our characterization of damage results in symmetric elastic-damage moduli. In the absence of further plastic flow, $\dot{\boldsymbol{\sigma}}^p \equiv \mathbf{0}$. Time differentiation of eqn (7) along with the damage rule (13a) and the damage consistency condition (14a) then yields

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, d) = (1-d)\frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} : \dot{\boldsymbol{\varepsilon}} - H(\bar{\tau}, d)\dot{\bar{\tau}}\boldsymbol{\sigma}^0 \tag{16}$$

where $\boldsymbol{\sigma}^0 := \partial \Psi^0(\boldsymbol{\varepsilon})/\partial \boldsymbol{\varepsilon}$ and, for notational simplicity, the subscript t has been omitted. We shall refer to $\boldsymbol{\sigma}^0$ as the *initial elastic stress*. By taking the time derivative of eqn (11) we obtain $\dot{\bar{\tau}} = (1/\bar{\tau})\boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}}$. Substitution into eqn (16) then yields $\dot{\boldsymbol{\sigma}} = \mathbf{C}(\boldsymbol{\varepsilon}, d) : \dot{\boldsymbol{\varepsilon}}$, where $\mathbf{C}(\boldsymbol{\varepsilon}, d)$ are the *elastic-damage tangent moduli* given by

$$\mathbf{C}(\boldsymbol{\varepsilon}, d) := \left[(1-d)\frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} - \frac{H}{\bar{\tau}}\boldsymbol{\sigma}^0 \otimes \boldsymbol{\sigma}^0\right]. \tag{17}$$

Note that $\mathbf{C}(\boldsymbol{\varepsilon}, d)$ is a *symmetric* rank four tensor. One typically assumes that the initial (undamaged) moduli $\mathbf{C}^0 := \partial^2 \Psi^0(\boldsymbol{\varepsilon})/\partial \boldsymbol{\varepsilon}^2$ are constant.

Remark 3.5. The symmetry of the elastic-damage moduli depends crucially on the form of $\dot{\bar{\tau}}$ in eqn (16) which follows from our definition of equivalent strain $\bar{\tau}$ given by eqn (11). In fact, the alternative definition $\bar{\tau} := \sqrt{(2J_2(\boldsymbol{\varepsilon}))} = \sqrt{(\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon})}$, suggested by Lemairte and Mazars[21, 22], would result in the following *non-symmetric* elastic-damage tangent moduli

$$\mathbf{C}(\boldsymbol{\varepsilon}, d) = \left[(1-d)\frac{\partial^2\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}^2} - \frac{H}{\bar{\tau}}\boldsymbol{\sigma}^0 \otimes \boldsymbol{\varepsilon}\right]. \quad \square \tag{18}$$

3.3. *Characterization of plastic response. Tangent moduli*

In accordance with the notion of effective stress, the characterization of the plastic response should be formulated in *effective stress space* in terms of effective stresses $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\sigma}}^p$. Thus, for the classical situation in which the yield function is postulated in stress space, we replace the homogenized Cauchy stress tensor $\boldsymbol{\sigma}$ by the *effective stress tensor* $\bar{\boldsymbol{\sigma}}$, so that the elastic-damage domain is characterized by $f(\bar{\boldsymbol{\sigma}}, \mathbf{q}) \leqslant 0$. Here, $\mathbf{q}$ are the internal plastic variables the evolution of which is defined below. With the assumption of an associative flow rule, rate-independent plastic response is characterized in strain space by the following constitutive equations

$$\begin{aligned}
\dot{\bar{\boldsymbol{\sigma}}}^p &= \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\varepsilon}}\left(\frac{\partial\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) && \textit{(associative flow rule)} \\
\dot{\mathbf{q}} &= \dot{\lambda}\mathbf{h}\left(\frac{\partial\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) && \textit{(plastic hardening law)} \\
& f\left(\frac{\partial\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) \leqslant 0 && \textit{(yield condition)}
\end{aligned} \tag{19}$$

where $\dot{\bar{\boldsymbol{\sigma}}}^p$ denotes the *plastic relaxation* effective stress rate tensor, $\dot{\lambda}$ denotes the plastic consistency parameter, and $\mathbf{h}$ signifies the hardening law. Equations (19) provide a characterization of plasticity in strain space (see Refs [38–41]). Loading/unloading conditions may be expressed in a compact form by requiring that

$$f\left(\frac{\partial\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) \leqslant 0, \qquad \dot{\lambda} \geqslant 0, \qquad \dot{\lambda} f\left(\frac{\partial\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) = 0. \tag{20}$$

Note that if $f < 0$ then $\dot{\lambda} = 0$ and the process is elastic-damage. On the other hand, for loading, $\dot{\lambda} > 0$ and $f = 0$. In this latter case, $\dot{\lambda}$ is determined by requiring that $\dot{f} = 0$, the so-called *plastic consistency condition*. Making use of the notation $\bar{\boldsymbol{\sigma}} \equiv (\partial\Psi^0(\boldsymbol{\varepsilon})/\partial\boldsymbol{\varepsilon}) - \bar{\boldsymbol{\sigma}}^p$ (see eqn (7)), during loading one has

$$\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} : \dot{\bar{\boldsymbol{\sigma}}} + \frac{\partial f}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} = 0 \tag{21}$$

where $\partial f/\partial \bar{\boldsymbol{\sigma}}$ denotes the partial derivative of $f((\partial\Psi^0(\boldsymbol{\varepsilon})/\partial\boldsymbol{\varepsilon}) - \bar{\boldsymbol{\sigma}}^p, \mathbf{q})$ with respect to the first argument. From eqn (7) we obtain

$$\dot{\bar{\boldsymbol{\sigma}}} = \frac{\partial^2\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}^2} : \dot{\boldsymbol{\varepsilon}} - \dot{\bar{\boldsymbol{\sigma}}}^p = \frac{\partial^2\Psi^0(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}^2} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right) \tag{22}$$

where use has been made of the flow rule $(19)_1$. Thus, $\dot{\lambda}$ is determined from eqns (21), (22), and the hardening law $(19)_2$ as

$$\dot{\lambda} = \frac{\dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} : \dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} - \dfrac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{h}}. \tag{23}$$

Substitution of eqn (23) into eqn (22) then yields $\dot{\bar{\boldsymbol{\sigma}}} = \bar{\mathbf{C}}^{ep} : \dot{\boldsymbol{\varepsilon}}$, where $\bar{\mathbf{C}}^{ep}$ are the *effective* elastoplastic tangent moduli given by

$$\bar{\mathbf{C}}^{ep} := \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} - \frac{\left[\dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} : \dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right] \otimes \left[\dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} : \dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right]}{\dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}^2} : \dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} - \dfrac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{h}}. \tag{24}$$

Since $\boldsymbol{\sigma} = (1-d)\bar{\boldsymbol{\sigma}}$, time differentiation and use of eqns (13a) and (14) along with the relation $\dot{\bar{\tau}} = \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}}/\bar{\tau}$ then leads to $\dot{\boldsymbol{\sigma}}^0 = \mathbf{C}^{ep} : \dot{\boldsymbol{\varepsilon}}$. Here $\mathbf{C}^{ep}$ are the elastoplastic-damage tangent moduli given by

$$\mathbf{C}^{ep} = (1-d)\bar{\mathbf{C}}^{ep} - \frac{H}{\bar{\tau}}[\bar{\boldsymbol{\sigma}} \otimes \boldsymbol{\sigma}^0]. \tag{25}$$

Observe from eqn (25) that $\mathbf{C}^{ep}$ is a non-symmetric *rank one update* of the symmetric tensor $(1-d)\bar{\mathbf{C}}^{ep}$.

Remark 3.6. According to eqn (19) plastic response is characterized in terms of *effective* stresses $\bar{\boldsymbol{\sigma}}^p$ and $\bar{\boldsymbol{\sigma}}$. From a physical standpoint, $\boldsymbol{\sigma}$ corresponds to a *homogenized* stress over a nominal area, whereas the effective stress $\bar{\boldsymbol{\sigma}}$ is a measure of the actual stress acting on the effective area. On this basis, a characterization of plastic flow in terms of effective stresses $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\sigma}}^p$ appears to be more appropriate. In addition, use of effective quantities in the yield condition has the net result of lowering the yield strength of the material. □

Remark 3.7 (*Alternative characterization*). Alternatively, one may characterize the plastic response of the material by postulating a flow rule that involves the *damaged* moduli according to the expression

$$\begin{aligned} &\dot{\boldsymbol{\sigma}}^p = \dot{\lambda}\mathbf{C}(\boldsymbol{\varepsilon}, d) : \frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \mathbf{q}, d) \qquad (\textit{flow rule}) \\ &\dot{\mathbf{q}} = \dot{\lambda}\mathbf{h}(\boldsymbol{\sigma}, \mathbf{q}, d) \qquad (\textit{plastic hardening law}) \\ &\dot{\lambda} \geqslant 0, \qquad f(\boldsymbol{\sigma}, \mathbf{q}, d) \leqslant 0, \qquad \dot{\lambda}\,(\boldsymbol{\sigma}, \mathbf{q}, d) = 0. \end{aligned} \tag{26}$$

Here $\mathbf{C}(\boldsymbol{\varepsilon}, d)$ denotes the elastic-damage moduli defined by eqn (17), and $(26)_3$ are the statement of the yield and loading/unloading conditions. During loading use of the plastic consistency condition $\dot{f}(\boldsymbol{\sigma}, \mathbf{q}, d) = 0$ leads, after standard arguments, to the following expression for elastoplastic-damage tangent moduli:

$$\mathbf{C}^{ep} := \mathbf{C} - \frac{\left[\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right] \otimes \left[\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right] + \dfrac{\partial f}{\partial d}\dfrac{H}{\bar{\tau}}\left[\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right] \otimes \boldsymbol{\sigma}^0}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{h}}. \tag{27}$$

Note that $\mathbf{C}^{ep}$ is generally non-symmetric. However, if one assumes that the plastic yield condition does not depend on the damage variable, then $\partial f/\partial d = \partial \mathbf{h}/\partial d = 0$, the second term in the numerator of eqn (27) drops out, and one obtains elastoplastic-damage moduli which are symmetric. It will be shown in Part II of this work that numerical modeling based on this formulation give satisfactory results. □

3.4. *Extension to rate-dependent (viscous) damage model*

Some existing experimental results (see, e.g. Refs [42–45]) suggest that the amount of microcracking (damage) at a particular strain level exhibits *rate sensitivity* to the applied rate of loading in a (high strain rate) dynamic environment. A simple phenomenological characterization of this effect may be obtained by means of a *viscous regularization* of the rate-independent strain-based damage model previously described. Formally, the structure of this regularization, formulated below, is entirely analogous to the classical viscoplastic regularization of the Perzyna type[46]. The resulting rate-sensitive damage model (a) requires only one additional material parameter, i.e. the *damage fluidity coefficient* (μ) ; (b) as μ approaches zero exhibits instantaneous elastic response, whereas as μ approaches infinity reduces to the rate-independent damage characterization ; and (c) predicts decrease in nonlinearity of the stress–strain curves as the strain rate is increased. In other words, microcrack growth is *retarded* at higher strain rates[45]. For alternative rate-sensitive damage theories see, e.g. Refs [42, 47]. In Ref [47], for instance, rate-sensitivity is constructed by considering the "microcrack inertia" effect.

Rate equations governing visco-damage behavior are obtained from their rate-independent counterpart (13a) by replacing the damage consistency parameter $\dot{\mu}$ by $\mu\bar{\phi}(g)$. Here μ is the damage fluidity coefficient (a material constant). The scalar function $\bar{\phi}(g)$ represents the viscous damage flow function and g is defined in eqn (12). Accordingly, we have

$$\begin{aligned} \dot{d}_t &= \mu\langle\bar{\phi}(g)\rangle H(\bar{\tau}_t, d_t) \\ \dot{r}_t &= \mu\langle\bar{\phi}(g)\rangle \end{aligned} \tag{28}$$

where $\langle\cdot\rangle$ denotes the McAuley bracket (ramp function). For simplicity, in what follows we shall assume *linear* viscous damage ; i.e. $\bar{\phi}(g) \equiv g$. Hence, eqns (28) reduce to

$$\begin{aligned} \dot{d}_t &= \mu\langle g(\bar{\tau}_t, r_t)\rangle H(\bar{\tau}_t, d_t) \\ \dot{r}_t &= \mu\langle g(\bar{\tau}_t, r_t)\rangle \equiv \mu\langle\bar{\tau}_t - r_t\rangle. \end{aligned} \tag{29}$$

The inviscid damage characterization and the instantaneous elasticity response can then be obtained as special cases of the rate-dependent damage formulation. These and the related algorithmic treatment of visco-damage will be addressed in detail in Part II of this work.

3.4.1. *Positive condition.* Recently, the question of uniqueness of solution and well-posedness of initial-value problems for softening materials has become a somewhat controversial issue. We show that a viscous damage model of the type (29) satisfies the *positiveness* condition in Valanis[34]. To this end, by differentiation of the stress–strain relation $\boldsymbol{\sigma} = (1-d)\nabla\Psi^0(\boldsymbol{\varepsilon})$ and using (29) we obtain

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= (1-d)\nabla^2\Psi^0(\boldsymbol{\varepsilon}):\dot{\boldsymbol{\varepsilon}} - \dot{d}\nabla\Psi^0(\boldsymbol{\varepsilon}) \\ &= (1-d)\nabla^2\Psi^0(\boldsymbol{\varepsilon}):\dot{\boldsymbol{\varepsilon}} - \mu\langle g(\bar{\tau}_t, r_t)\rangle H(\bar{\tau}_t, d_t)\nabla\Psi^0(\boldsymbol{\varepsilon}). \end{aligned} \tag{30}$$

At a state defined by $\{\boldsymbol{\varepsilon}, d_t, r_t\}$, for two different stress rates $\dot{\boldsymbol{\sigma}}_1$, $\dot{\boldsymbol{\sigma}}_2$, and two different strain rates $\dot{\boldsymbol{\varepsilon}}_1$ and $\dot{\boldsymbol{\varepsilon}}_2$, it follows from eqn (30) that

$$(\dot{\boldsymbol{\sigma}}_1 - \dot{\boldsymbol{\sigma}}_2):(\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\varepsilon}}_2) = (1-d)\,(\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\varepsilon}}_2):\nabla^2\Psi^0(\boldsymbol{\varepsilon}):\nabla^2\Psi^0(\boldsymbol{\varepsilon}):(\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\varepsilon}}_2) > 0 \tag{31}$$

provided that the undamaged elastic modulus $\mathbf{C}^0 := \nabla^2\Psi^0(\boldsymbol{\varepsilon})$ is positive definite. Thus, the material is *positive* in the sense of Valanis[34].

4. A SIMPLE STRAIN-BASED ANISOTROPIC DAMAGE MODEL

In this section we shall extend the *strain*-based framework developed in Section 3 to include *anisotropic brittle* damage. It is shown that for ductile damage the characterization proposed in this section collapses to the isotropic model developed in Section 3, and thus providing a canonical extension of ductile (isotropic) damage models to treat brittle anisotropic damage.

Anisotropic continuum damage may be characterized in terms of vector quantities[16, 17], or by means of second- or fourth-order tensorial quantities[25, 26, 32, 33, 48]. In the present context, we propose a characterization of damage based on the concept of effective stress, by providing a simple and effective construction of the fourth-order transformation tensor $\mathbf{M}$. We recall that this transformation connects $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$ through the basic relation (1). We start by assuming the following form of free energy potential

$$\Psi := \tfrac{1}{2}\boldsymbol{\varepsilon}:\mathbf{C}:\boldsymbol{\varepsilon}-\boldsymbol{\sigma}^{\mathrm{p}}:\boldsymbol{\varepsilon}+\Xi(\mathbf{q}) \tag{32}$$

in which $\mathbf{C}$ is a fourth-order tensor that physically defines the current *damaged* stiffness moduli, and thus includes the effect of microcracking. The Clausius–Duhem inequality (6) now yields

$$\dot{\boldsymbol{\varepsilon}}:[\boldsymbol{\sigma}-\mathbf{C}:\boldsymbol{\varepsilon}+\boldsymbol{\sigma}^{\mathrm{p}}]-\boldsymbol{\varepsilon}:\dot{\mathbf{C}}:\boldsymbol{\varepsilon}+[\dot{\boldsymbol{\sigma}}^{\mathrm{p}}:\boldsymbol{\varepsilon}-\nabla\Xi\cdot\dot{\mathbf{q}}] \geqslant 0. \tag{33}$$

Note that $\mathbf{C}$ plays itself the role of the internal damage variable. In Ref. [48] the compliance tensor is also regarded as an internal variable. As in Section 2 we assume that unloading is an elastic process together with the hypothesis that damage evolution (characterized below) and plastic evolution are independent processes. This leads to the stress–strain relations

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = \mathbf{C}:\boldsymbol{\varepsilon}-\boldsymbol{\sigma}^{\mathrm{p}} \tag{34}$$

along with the following damage and plastic dissipation inequalities

$$D^{\mathrm{d}} := -\boldsymbol{\varepsilon}:\dot{\mathbf{C}}:\boldsymbol{\varepsilon} \geqslant 0, \qquad D^{\mathrm{p}} := \dot{\boldsymbol{\sigma}}^{\mathrm{p}}:\boldsymbol{\varepsilon}-\nabla\Xi\cdot\dot{\mathbf{q}} \geqslant 0. \tag{35}$$

In addition, we note that

$$\frac{\partial\Psi}{\partial(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})} = \frac{1}{2}\mathbf{C} \tag{36}$$

so that $(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})$ is the thermodynamic flux conjugate to the (tensor) internal variable $\mathbf{C}$.

Remark 4.1. Let us denote by $\mathbf{C}^0$ the initial *undamaged* moduli at time $t = 0$; that is, $\mathbf{C}^0 := \mathbf{C}|_{t=0}$. Within the present framework we can introduce the notion of *effective stress* by setting

$$\mathbf{M} := \mathbf{C}\mathbf{C}^{0-1}, \qquad \bar{\boldsymbol{\sigma}} := \mathbf{M}^{-1}:\boldsymbol{\sigma}, \qquad \bar{\boldsymbol{\sigma}}^{\mathrm{p}} := \mathbf{M}^{-1}:\boldsymbol{\sigma}^{\mathrm{p}}. \tag{37}$$

Equation (34) may then be rephrased in terms of effective stresses and undamaged moduli as

$$\bar{\boldsymbol{\sigma}} = \mathbf{C}^0:\boldsymbol{\varepsilon}-\bar{\boldsymbol{\sigma}}^{\mathrm{p}} \tag{38}$$

which is the counterpart of eqn (7). Therefore, plastic response can be characterized independently from damage evolution in terms of effective stress exactly as in Section 3. □

The main point of the present development concerns the characterization of damage.

4.1. *Characterization of damage evolution*

In order to build into the formulation the notion of *irreversibility*, we introduce a damage criterion in *strain space* with the following functional form

$$g(\varepsilon \otimes \varepsilon, r_t) := G(\varepsilon \otimes \varepsilon) - r_t \leqslant 0 \tag{39}$$

where r_t is an internal variable that furnishes the "radius" of the damage surface $g(\varepsilon \otimes \varepsilon, r_t) = 0$ at current time. The damage process is then characterized in terms of the following irreversible, dissipative equations of evolution

$$\begin{gathered}\dot{\mathbf{C}} = -\dot{\mu}\frac{\partial g(\varepsilon \otimes \varepsilon, r_t)}{\partial(\varepsilon \otimes \varepsilon)} \\ \dot{\mu} \geqslant 0, \qquad g(\varepsilon \otimes \varepsilon, r_t) \leqslant 0, \qquad \dot{\mu} g(\varepsilon \otimes \varepsilon, r_t) \equiv 0.\end{gathered} \tag{40}$$

Equations (40) can be regarded as the Kuhn–Tucker conditions of the following "principle of maximum damage dissipation": ***For a given local history of strains the actual damage moduli* C *are those moduli that render a maximum of damage dissipation.*** This principle is analogous to the principle of maximum plastic dissipation (see e.g. Hill[49]). A proof of this statement is sketched below.

Remark 4.2. Mathematically, maximum damage dissipation may be formulated as follows. Introduce the convex cone

$$E := \{\varepsilon \in [L^2(\Omega)]^6 \mid g(\varepsilon \otimes \varepsilon, r_t) \leqslant 0\}. \tag{41}$$

Then, the moduli $\mathbf{C}$ are characterized as the argument of the following principle

$$\mathbf{C} = \arg\max_{\varepsilon \in E} \{D^{\mathrm{d}} := -\varepsilon : \dot{\mathbf{C}} : \varepsilon\}. \tag{42}$$

To see this, simply note that eqns (40) are the Kuhn–Tucker optimality conditions (see, e.g. Strang[50]) of the following Lagrangian functional

$$L^{\mathrm{d}} := D^{\mathrm{d}} - \dot{\mu}\, g(\varepsilon \otimes \varepsilon, r_t) \tag{43}$$

where $\dot{\mu} \geqslant 0$ is a Lagrange multiplier belonging to the positive cone $K := \{\dot{\mu} \in L^2(\Omega) \mid \dot{\mu} \geqslant 0\}$ (the damage rule, eqn $(40)_1$, corresponds to requiring that $\partial L^{\mathrm{d}}/\partial(\varepsilon \otimes \varepsilon) = \mathbf{0}$). □

4.2. *Ductile and brittle damage models*

We show next that for the case of ductile damage the characterization outlined above reduces to the isotropic damage model described in Section 3. By contrast, for the brittle damage case, the above formulation together with a positive spectral projection results in an anisotropic damage model.

4.2.1. *Ductile damage: isotropy.* Assume that the damage rule is of the form

$$g(\varepsilon \otimes \varepsilon, r_t) := G(\bar{\tau}) - r_t, \qquad \text{where } \bar{\tau} := \sqrt{(\varepsilon : \mathbf{C}^0 : \varepsilon)}. \tag{44}$$

That is, $\bar{\tau}$ is the equivalent strain concept introduced in Section 3. In addition, we define $H := \partial G/\partial \bar{\tau}$ and $\dot{r}_t \equiv \dot{\mu}H/2\bar{\tau}$. Since $\dot{r}_t = \dot{\bar{\tau}}H$, we have $\dot{\mu} = 2\bar{\tau}\dot{\bar{\tau}}$. In addition, $\partial\bar{\tau}/\partial(\varepsilon \otimes \varepsilon) = (1/2\bar{\tau})\mathbf{C}^0$ and the damage rule $(40)_1$ thus takes the form

$$\dot{\mathbf{C}} = -\dot{\bar{\tau}}H\mathbf{C}^0 \Rightarrow \mathbf{C} = (1-d)\mathbf{C}^0 \tag{45}$$

where, by damage consistency

$$\dot{d} = \frac{\partial G(\bar{\tau})}{\partial \bar{\tau}} \dot{\bar{\tau}} = \frac{H}{\bar{\tau}} \boldsymbol{\varepsilon} : \mathbf{C}^0 : \dot{\boldsymbol{\varepsilon}}. \tag{46}$$

Therefore, in the event of ductile damage, the above characterization implies that damage *must be* isotropic.

4.2.2. *Brittle strain-based anisotropic damage*. The proposed formulation of damage outlined above leads to a very simple anisotropic brittle damage model. First, in view of the significance of tensile extensions in brittle damage processes, we propose the following definition for the *equivalent tensile strain*. Consider the spectral decomposition of the strain tensor[21, 22, 48]

$$\boldsymbol{\varepsilon} = \sum_{i=1}^{3} \varepsilon_i \mathbf{p}_i \otimes \mathbf{p}_i, \qquad \|\mathbf{p}_i\| = 1 \tag{47}$$

where ε_i is the ith principal strain and $\mathbf{p}_i$ the ith corresponding unit principal direction. Let $\mathbf{Q}^+$ be the *positive* spectral projection defined as

$$\mathbf{Q}^+ := \sum_{i=1}^{3} \bar{H}(\varepsilon_i)\mathbf{p}_i \otimes \mathbf{p}_i, \qquad \|\mathbf{p}_i\| = 1 \tag{48}$$

where $\bar{H}(\cdot)$ is the Heaviside function. Then, we define the *tensile strain tensor* $\boldsymbol{\varepsilon}^+$ by the expression

$$\boldsymbol{\varepsilon}^+ := (\mathbf{Q}^+)\,\mathrm{diag}\,[\varepsilon_1, \varepsilon_2, \varepsilon_3]\,(\mathbf{Q}^+)^T \equiv [\mathbf{Q}^+\mathbf{Q}^T]\boldsymbol{\varepsilon}[\mathbf{Q}^+\mathbf{Q}^T]^T \tag{49}$$

where $\mathbf{Q} := \sum_{i=1}^{3} \mathbf{p}_i \otimes \mathbf{p}_i$ and tensile strains are taken to be positive. For convenience we introduce the fourth-order projection tensor $\mathbf{P}^+$ with components

$$P^+_{ijkl} = Q^+_{ia}\, Q^+_{jb}\, Q_{ka}\, Q_{lb} \tag{50}$$

so that ε^+ can be expressed as

$$\boldsymbol{\varepsilon}^+ = \mathbf{P}^+ : \boldsymbol{\varepsilon}, \qquad \text{i.e. } \varepsilon^+_{ij} = P^+_{ijkl}\varepsilon_{kl}. \tag{51}$$

With this notation at hand, we introduce the notion of *equivalent tensile strain* $\bar{\tau}$ according to the expression

$$\bar{\tau} := \sqrt{(\boldsymbol{\varepsilon}^+ : \mathbf{C}^0 : \boldsymbol{\varepsilon}^+)} \equiv \sqrt{(\boldsymbol{\varepsilon} : [\mathbf{P}^+\mathbf{C}^0\mathbf{P}^+] : \boldsymbol{\varepsilon})}. \tag{52}$$

The damage rule then becomes

$$\begin{aligned} \dot{\mathbf{C}} &= -\dot{\bar{\tau}} H \mathbf{P}^+\mathbf{C}^0\mathbf{P}^+ \\ \dot{\bar{\tau}} &= \frac{1}{\bar{\tau}}\boldsymbol{\varepsilon}^+ : \mathbf{C}^0 : \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\varepsilon}^+) \\ \boldsymbol{\varepsilon}^+ &= \mathbf{P}^+ : \boldsymbol{\varepsilon} \end{aligned} \tag{53}$$

where use has been made of the relationship $2\bar{\tau}\dot{\bar{\tau}} \equiv \dot{\mu}$ in eqn $(53)_1$. This completes our formulation of brittle damage. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P}^+ : \boldsymbol{\varepsilon}) \neq \mathbf{P}^+ : \dot{\boldsymbol{\varepsilon}}$$

owing to the *non-linear* nature of $\mathbf{P}^+$.

Remark 4.3. The actual computation of the projection $\mathbf{P}^+$ as defined by eqn (50) involves an eigenvalue calculation to obtain $\mathbf{Q}^+$ given eqn (48). A further simplification is obtained by selecting $\mathbf{P}^+$ as the *volumetric* tensile strain projection. Accordingly, one could set

$$\mathbf{P}^+ := \tfrac{1}{3}\bar{H}(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{1} \otimes \mathbf{1}, \qquad \boldsymbol{\varepsilon}^+ = \mathbf{P}^+ : \boldsymbol{\varepsilon} \tag{54}$$

where tensile strain is taken to be positive. Note for the isotropic case the damage evolution equation, eqn (53), amounts to degradation of the bulk modulus. □

4.2.3. *Physical motivation*. The brittle damage model outlined above is essentially a *strain driven* mechanism in the sense that damage in the material is directly linked to the history of strains and not to the stress history. To illustrate the physical implications of this basic assumption, consider the idealized situation of a cylinder subject to unconfined increasing uniaxial compression. Assuming isotropic undamaged elastic response, a strain-based anisotropic brittle damage model would predict progressive damage and eventual failure of the specimen due to the presence of *tensile* radial and hoop strains. Cracking then develops normal to the plane of tensile strains and thus parallel to the axis of loading; i.e. the so-called "*splitting modes*". This is a typical failure in many rock-like materials such as concrete. Note that a phenomenological damage model based on tensile stresses could not possibly predict such a failure mode. □

From a computational standpoint, a strain-based damage criterion is particularly convenient and, as shown in Part II of this work, leads to a remarkably simple algorithmic treatment.

5. A STRESS-BASED ISOTROPIC CONTINUUM DAMAGE MODEL

In this section we develop a stress-based characterization of elastic–plastic damage response which is *dual* to the strain-based formulation considered in Section 3. Here, we start from an assumed form of the complementary energy function and obtain, by systematically exploiting the Clausius–Duhem inequality, the notion of effective strain, the additive split of the strain tensor into elastic and plastic parts, and the hypothesis of stress equivalence as formulated in Section 2.2. Although stress- and strain-based characterization of damage may be regarded as *dual* points of view, they are not equivalent neither physically nor computationally. We finally recall that fracture criteria used in fracture mechanics are typically stress based.

5.1. *Thermodynamic basis. Strain split*

We start our formulation by considering a complimentary free energy potential of the following form

$$\Lambda(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \mathbf{q}, d_\sigma) := d_\sigma \Lambda^0(\boldsymbol{\sigma}) + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{\mathrm{p}} - \Xi(\mathbf{q}, \boldsymbol{\varepsilon}^{\mathrm{p}}) \tag{55}$$

where $d_\sigma := 1/(1-d)$, $\boldsymbol{\varepsilon}^{\mathrm{p}}$ is the *plastic* strain tensor, and $\Lambda^0(\boldsymbol{\sigma})$ is the *initial elastic* complimentary stored energy function of the virgin material. For an isothermal case, the Clausius–Duhem inequality (7) expressed in terms of the complementary free energy takes the form

$$\dot{\Lambda} - \dot{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} \geqslant 0 \tag{56}$$

for any admissible process. Time differentiation of eqn (55), substitution of the result into inequality (56), and use of standard arguments together with the additional assumption that plastic and damage unloading are elastic processes (in agreement with the characterizations discussed below), yields

$$\boldsymbol{\varepsilon} = \frac{\partial \Lambda}{\partial \boldsymbol{\sigma}} = d_\sigma \frac{\partial \Lambda^0}{\partial \boldsymbol{\sigma}} + \boldsymbol{\varepsilon}^{\mathrm{p}} \tag{57}$$

along with the dissipative inequalities

$$\Lambda^0(\boldsymbol{\sigma})\dot{d}_\sigma \geqslant 0 \qquad \text{and} \qquad -\frac{\partial \Xi}{\partial \mathbf{q}}\cdot\dot{\mathbf{q}} - \left(\frac{\partial \Xi}{\partial \boldsymbol{\varepsilon}^{\mathrm{p}}} - \boldsymbol{\sigma}\right):\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geqslant 0. \tag{58}$$

One observes that decomposition, eqn (57), for the strain tensor is the counterpart of eqn (7) for the stress tensor. Moreover, from eqns (55) and $(58)_1$ it also follows that

$$-Y := -\frac{\partial \Lambda(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \mathbf{q}, d_\sigma)}{\partial d_\sigma} = \Lambda^0(\boldsymbol{\sigma}). \tag{59}$$

Hence, the *initial elastic* complimentary energy $\Lambda^0(\boldsymbol{\sigma})$ is the thermodynamic force $-Y$ *conjugate* to the damage variable d_σ. This observation motivates the characterization of damage in Section 5.2.

Remark 5.1. We assume that the complementary energy function $\Lambda^0(\boldsymbol{\sigma})$ is such that

$$\Lambda^0(\mathbf{0}) = 0, \qquad \text{and} \qquad \left.\frac{\partial \Lambda^0(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right|_{\boldsymbol{\sigma}=\mathbf{0}} = \mathbf{0}. \tag{60}$$

Note that for the linear case $\Lambda^0 := 1/2\boldsymbol{\sigma} : \mathbf{C}^{0-1} : \boldsymbol{\sigma}$, where $\mathbf{C}^0$ are the undamaged elastic moduli. It follows from eqns (57) and (60) that the *plastic strain* $\boldsymbol{\varepsilon}^{\mathrm{p}}$ is precisely the residual strain obtained upon (local) unloading. Thus, identifying the elastic strain with *recoverable* strain after unloading, i.e. $\boldsymbol{\varepsilon}^{\mathrm{e}} := \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}$, from eqn (57) we have

$$\boldsymbol{\varepsilon}^{\mathrm{e}} \equiv d_\sigma \frac{\partial \Lambda^0}{\partial \boldsymbol{\sigma}} \Rightarrow \bar{\boldsymbol{\varepsilon}}^{\mathrm{e}} := (1-d)\boldsymbol{\varepsilon}^{\mathrm{e}} = \frac{\partial \Lambda^0}{\partial \boldsymbol{\sigma}} \tag{61}$$

since $d_\sigma = 1/(1-d)$. □

5.2. *Stress-based characterization of damage. Elastic-damage moduli*

By analogy with the treatment in Section 3.2, we characterize evolution of damage in the material by means of a *damage criterion* and a damage rule. First, motivated by the conjugacy relation (59) we define the *equivalent strain* as the (undamaged) complimentary energy norm of the stress tensor

$$\bar{\tau} := \sqrt{(2\Lambda^0(\boldsymbol{\sigma}))}. \tag{62}$$

Next, we postulate a damage criterion $g(\bar{\tau}_t, r_t) \leqslant 0$, formulated in stress space, with the following functional form

$$g(\bar{\tau}, r) := \bar{\tau} - r \leqslant 0. \tag{63}$$

Condition (63) states that damage in the material is initiated when the *complimentary energy norm* of the stress tensor, $\bar{\tau}$, exceeds the initial damage threshold r_0. For the isotropic case, we define the evolution of the damage variable d_σ and the damage threshold r_t according to the rate equations

$$\dot{d}_\sigma = \dot{\mu} H(d_\sigma, \bar{\tau}) \qquad \dot{r} = \dot{\mu} \tag{64a}$$

together with the *damage loading/unloading* conditions

$$\dot{\mu} \geqslant 0, \qquad g(\bar{\tau}, r) \leqslant 0, \qquad \dot{\mu} g(\bar{\tau}, r) = 0. \tag{64b}$$

The value of $\dot{\mu}$ is determined by the damage consistency condition; i.e.

$$g(\bar{\tau}, r) = \dot{g}(\bar{\tau}, r) = 0 \Rightarrow \dot{\mu} = \dot{\bar{\tau}}. \tag{65}$$

This leads to the explicit expression: $r = \max\{r_0, \max_{s \in (-\infty, t]} \bar{\tau}_s\}$.

5.2.1. *Elastic-damage tangent moduli.* Consider the situation in which no further plastic flow takes place so that $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$. Time differentiation of eqn (57) then yields

$$\dot{\boldsymbol{\varepsilon}} = \dot{d}_\sigma \frac{\partial \Lambda^0}{\partial \boldsymbol{\sigma}} + d_\sigma \frac{\partial^2 \Lambda^0}{\partial \boldsymbol{\sigma}^2} : \dot{\boldsymbol{\sigma}}. \tag{66}$$

Again we recall that $d_\sigma := 1/(1-d)$. Assuming that the material is undergoing further damage, the Kuhn–Tucker conditions (64b) yield $\dot{\mu} > 0$ and $g(\bar{\tau}, r) = 0$. By enforcing the damage consistency condition $\dot{g}(\bar{\tau}, r) = 0$ we obtain $\dot{\mu} \equiv \dot{\bar{\tau}} = (1/\bar{\tau})\,(\partial \Lambda^0/\partial \boldsymbol{\sigma}) : \dot{\boldsymbol{\sigma}}$. Therefore, $\dot{\boldsymbol{\varepsilon}} = \mathbf{C}^{-1}(\boldsymbol{\sigma}, d_\sigma) : \dot{\boldsymbol{\sigma}}$, where $\mathbf{C}(\boldsymbol{\sigma}, d_\sigma)$ are the *elastic-damage tangent moduli* given by

$$\mathbf{C}(\boldsymbol{\sigma}, d_\sigma) := \left[d_\sigma \frac{\partial^2 \Lambda^0(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}^2} + \frac{H}{\bar{\tau}} \frac{\partial \Lambda^0}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \Lambda^0}{\partial \boldsymbol{\sigma}} \right]^{-1}. \tag{67}$$

Note that $\mathbf{C}(\boldsymbol{\sigma}, d_\sigma)$ is a *symmetric* rank four tensor.

5.3. *Characterization of plastic response. Tangent moduli*

Within the present stress-based framework, we characterize general plastic response in the stress space by means of the classical constitutive equations

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \mathbf{q}) && (\textit{associative flow rule}) \\ \dot{\mathbf{q}} &= \dot{\lambda} \mathbf{h}(\boldsymbol{\sigma}, \mathbf{q}) && (\textit{plastic hardening law}) \\ f(\boldsymbol{\sigma}, \mathbf{q}) &\leqslant 0 && (\textit{yield condition}). \end{aligned} \tag{68}$$

In addition, loading/unloading conditions may be conveniently formulated in Kuhn–Tucker form as

$$f(\boldsymbol{\sigma}, \mathbf{q}) \leqslant 0, \qquad \dot{\lambda} \geqslant 0, \qquad \dot{\lambda} f(\boldsymbol{\sigma}, \mathbf{q}) = 0. \tag{69}$$

Finally, elastoplastic-damage tangent moduli are obtained under the conditions $\dot{\lambda} > 0$ and $\dot{\mu} > 0$, by enforcing both the plastic and damage consistency conditions $\dot{f} = 0$ and $\dot{g} = 0$ during loading. Since $\dot{\boldsymbol{\varepsilon}}^p \neq 0$, an argument similar to that leading to eqn (66) now yields

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) \tag{70}$$

where $\mathbf{C}$ is the elastic-damage modulus defined by eqn (67), and use has been made of flow rule $(68)_1$. $\dot{\lambda}$ is determined from the consistency condition $\dot{f}(\boldsymbol{\sigma}, \mathbf{q}) = 0$ as

$$\dot{\lambda} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{h}}. \tag{71}$$

We then obtain the following expression for the elastoplastic-damage tangent moduli

$$\mathbf{C}^{\mathrm{ep}} := \mathbf{C} - \frac{\left[\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right] \otimes \left[\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right]}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} - \dfrac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{h}}. \tag{72}$$

If plastic loading is taking place but the material is not undergoing further damage; i.e. $\dot{\lambda} > 0, f = 0$ but $\dot{\mu} = 0$, we arrive at expressions for $\dot{\lambda}$ and $\mathbf{C}^{\mathrm{ep}}$ identical to eqns (70) and (71) with $\mathbf{C}$ now given by

$$\mathbf{C} = \left[d_\sigma \frac{\partial^2 \Lambda^0}{\partial \boldsymbol{\sigma}^2} \right]^{-1}.$$

Note that $\mathbf{C}^{\mathrm{ep}}$ is a *symmetric* rank four tensor. The reason for this symmetry lies in the formulation of the plastic flow rule for the plastic strain $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, not for the effective plastic strain $\dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{p}}$.

Remark 5.2. Note that according to eqn (72) the elastoplastic-damage moduli have a compelling physical interpretation: their structure is identical to the classical elastoplastic moduli, with the elastic moduli of the undamaged material replaced by the elastic-damage moduli associated with the damaged material. □

Remark 5.3. One can extend the above stress-based damage model to account for brittle behavior. The formulation proceeds along the lines developed in Section 4. □

6. APPLICATION: A CAP-DAMAGE MODEL FOR CONCRETE

Concrete is known to behave as a brittle material that contains numerous microcracks and microvoids. From experimental observations, damage in concrete is a continuous process that initiates at very low levels of the applied loading[51, 52], with increasing amount of damage for increasing levels of strain. The amount of damage that takes place at very low strain or stress levels may be considered insignificant[51]. Essentially, significant damage appears only beyond a certain strain threshold[20–22]. In addition, the so-called *mode I* behavior (tensile cracking) is observed to be the dominant phenomenological aspect in the concrete damage process[21].

In this section, a cap plasticity model with an isotropic strain-based damage mechanism is developed to capture basic features of the behavior of concrete, by specialization of our formulation. To this end, the inviscid, two-invariant associative cap model originally proposed by DiMaggio and Sandler[53–55] is taken as a point of departure. We conclude this section with a viscoplastic extension for the proposed cap-damage model to account for rate effects. It is noted that we adopt the effective stress concept and the strain-based damage model in Section 3 in the following developments.

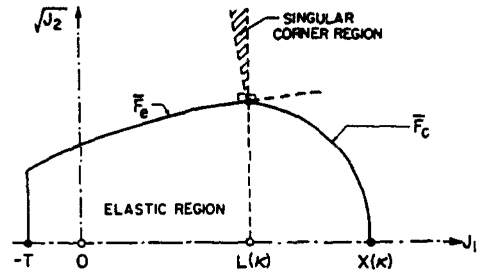

Fig. 3. The yield surface of the two-invariant cap model in effective pressure/J_2-deviator space. $\bar{F}_e$ and $\bar{F}_c$ denote the failure envelope and the hardening cap surface, respectively. The shaded area is the "singular corner region".

6.1. *Summary of the model*

It is assumed that the model exhibits isotropic response of the form

$$\boldsymbol{\sigma} = (1-d)\,[K\,(\mathrm{tr}\,\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\,\mathbf{e}] - \boldsymbol{\sigma}^{\mathrm{p}} \tag{73}$$

where $\mathbf{e} := \boldsymbol{\varepsilon} - 1/3\ (\mathrm{tr}\,\boldsymbol{\varepsilon})\ \mathbf{1}$ is the strain deviator. The basic characteristic of the cap model is the form of the yield function $f(\bar{\boldsymbol{\sigma}}, \kappa)$ which is specified in terms of two functions $\bar{F}_e$ and $\bar{F}_c$. The function $\bar{F}_e$ denotes the so-called *failure envelope surface* whereas the function F_c is referred to as the *hardening cap*. Functional forms for $\bar{F}_e$ and $\bar{F}_c$ are (see Fig. 3)

$$f(\bar{\boldsymbol{\sigma}}, \kappa) := \begin{cases} \bar{J}_2 - \bar{F}_e(\bar{J}_1) \leqslant 0 & \textit{(failure envelope)} \\ \bar{J}_2 - \bar{F}_c(\bar{J}_1, L(\kappa)) \leqslant 0 & \textit{(cap surface)} \end{cases} \tag{74a}$$

where $\bar{J}_1 := \mathrm{tr}\,\bar{\boldsymbol{\sigma}}, \bar{J}_2 := \frac{1}{2}\,\bar{\mathbf{s}} : \bar{\mathbf{s}}$, and

$$\bar{F}_e(\bar{J}_1) := [F_e(\bar{J}_1)]^2\,; F_e(\bar{J}_1) := [\alpha - \gamma \exp(-\beta \bar{J}_1) + \Theta \bar{J}_1]$$

$$L(\kappa) := \langle \kappa \rangle = \begin{cases} \kappa & \text{if } \kappa > 0 \\ 0 & \text{if } \kappa \leqslant 0 \end{cases} \tag{74b}$$

$$\bar{F}_c(\bar{J}_1, L(\kappa)) := \bar{F}_e(L(\kappa)) - \frac{[\bar{J}_1 - L(\kappa)]^2}{R^2}.$$

Here $\langle \cdot \rangle$ is referred to as the McAuley bracket. According to Remark 3.2, the plastic volume change $\varepsilon_v^{\mathrm{p}} := \mathrm{tr}\,\boldsymbol{\varepsilon}^{\mathrm{p}}$ is obtained from eqns (10) and (73) as

$$\varepsilon_v^{\mathrm{p}} := \frac{\mathrm{tr}\,\boldsymbol{\sigma}^{\mathrm{p}}}{3(1-d)K}.$$

In the cap model the *hardening parameter* κ is related to the plastic volume change $\varepsilon_v^{\mathrm{p}}$ by the hardening law

$$\varepsilon_v^{\mathrm{p}}(X) := W\{1 - \exp[-DX(\kappa)]\} = h(\kappa) \tag{75}$$

$$X(\kappa) := \kappa + RF_e(\kappa) \tag{76}$$

In the above expressions, α, β, γ, Θ, R, D and W are material parameters. To account for damage effects, we assume that $H(\bar{\tau}, d)$ in the equation of evolution, eqn (13a), for $\dot{d}$ is independent of d. Then, according to Remark 3.3, we introduce a damage accumulation function $G(\bar{\tau})$, defined by eqns (15), with the following functional form proposed in Refs [21, 22] for concrete material

$$G(\bar{\tau}) := 1 - \frac{\bar{\tau}_0(1-A)}{\bar{\tau}} - A \exp[B(\bar{\tau}_0 - \bar{\tau})] \tag{77}$$

where A and B are characteristic material parameters and $\bar{\tau}_0$ denotes the characteristic equivalent strain corresponding to the initial damage threshold r_0. As shown in Part II of this paper, these parameters can be estimated in a systematic manner from suitable experimental data.

Remark 6.1. The cap model outlined above corresponds to the commonly accepted interpretation of the original DiMaggio–Sandler model. There are, however, alternative formulations. For a discussion of all possible alternative formulations along with a complete analysis, we refer to Simo *et al.*[60]. □

6.2. *Viscoplastic (rate-dependent) extension*

Within the framework of a viscoplastic formulation of the type proposed by Perzyna[46], the rate effect can be readily accommodated in the inviscid cap-damage model developed above. Following Ref. [56], we assume loading surfaces with identical functional form as the yield surfaces in the inviscid case. We postulate an associative viscoplastic flow rule and a hardening law of the following form

$$\begin{aligned} \dot{\bar{\boldsymbol{\sigma}}}^{vp} &= \frac{1}{\tau}\langle\phi(f)\rangle\frac{\partial f}{\partial \boldsymbol{\varepsilon}}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) \\ \dot{\mathbf{q}} &= \frac{1}{\tau}\langle\phi(f)\rangle\mathbf{h}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}^p, \mathbf{q}\right) \end{aligned} \tag{78}$$

where τ is the relaxation time (viscosity coefficient). In addition, $\phi(f)$ denotes the viscous flow, a dimensionless scalar function, and f is the viscoplastic loading function. It is clear that the constitutive relations governing viscoplastic behavior are obtained from their inviscid counterpart, eqns (19), simply by replacing $\dot{\lambda}$ by $\langle\phi(f)\rangle/\tau$. Two commonly assumed forms of the viscoplastic flow function for a two-invariant isotropic constitutive model are

$$\phi(f) = \left(\frac{f}{|\bar{F}|}\right)^N \quad \text{or} \quad \phi(f) = \exp\left(\frac{f}{|\bar{F}|}\right)^N - 1 \tag{79}$$

where $N \in \mathbb{R}_+$ and $\bar{F}$ signifies $\bar{F}_c$ or $\bar{F}_c$ (see eqns (74a) and (74b)).

In Part II of this paper it is shown that despite the simplicity of the cap-damage model outlined above, good agreement is obtained with well-documented experimental data for concrete. In particular, softening behavior is well captured. The numerical implementation of the model is also discussed in detail in Part II of this paper.

7. CLOSURE

We close this paper by noting that for brittle damage, the proposed strain-based anisotropic characterization of damage in terms of the strain history is capable of predicting failure mode in which cracking develops parallel to the axis of loading; i.e. splitting modes. In addition, the proposed strain-based equivalent strain concept defined as the initial strain energy norm of the undamaged material results, for the ductile damage case, in symmetric elastic-damage moduli.

General plastic response is introduced either by means of an additive split of the stress tensor into elastic-damage and plastic relaxation parts in a strain-based model, or by means of an additive split of the strain tensor into elastic and plastic parts in a stress-based model. These decompositions result from the assumed forms of free energy potential. To illustrate the basic formulation, we have developed a simple model for concrete by incorporating an isotropic damage mechanism into the two-invariant cap plasticity model. Rate-dependent effects are accommodated in this model by means of a viscoplastic regularization of the Perzyna type.

A basic purpose of the present work is to demonstrate that the proposed classes of elastoplatic-damage constitutive equations are particularly well suited for large-scale computation. In Part II of this paper an efficient class of unconditionally stable integration algorithms is developed, based on the notion of operator splitting. Numerical simulations that illustrate the formulation developed herein are presented and discussed in detail in Part II of this work.

Acknowledgements—We are indebted to Professors K. S. Pister and R. L. Taylor for many helpful discussions. This work was sponsored by the Defense Nuclear Agency under Contract No. DNA-2DJA739 with Stanford University, and Contract No. DNA001-84-C-0304 with the University of California, Berkeley. This support and the interest and comments of Dr Eugene Sevin are gratefully acknowledged.

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